Relativistic Formulation of Maxwell's Equations for Free Space

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Einstein assumed in his Special Theory of Relativity that Maxwell's equations, including Faraday's law and the Ampere-Maxwell equation, were invariant in any inertial frame, and that the Lorentz transformation equations must be used when two inertial frames were in relative motion. Starting with a modification of the Ampere-Maxwell equation that allows for two observers of the magnetic field in different inertial frames, I offer an alternative formulation of Maxwell's wave equations for free space. The modification is based on two equal but different definitions of the speed of light. One definition relates to the particle-like properties of light and the other relates to the wave-like properties of light. The proposed formulation is consistent with the two postulates of the Special Theory of Relativity. The resulting equations, which are invariant in any inertial frame and are based on Euclidean space and Newtonian time, do not require the Lorentz transformations. The resulting equations allow for anisotropy in the electromagnetic waves that leads to an anisotropy in the Poynting vector that is able to act on a particle with a charge and/or a magnetic moment moving through a radiation field. The anisotropy of the Poynting vector results in radiation friction that opposes the movement of the particle and limits the velocity of the particle to the speed of light.

1. Introduction

Maxwell [1] developed his electrodynamic equations in terms of absolute Euclidean space and Newtonian time. Lorentz [2,3] assumed that Maxwell's equations were true only in the inertial frame of the ether that was characterized solely by its electric permittivity and magnetic permeability. Lorentz further assumed that the electrodynamic and optical phenomena existed in absolute Euclidean space and Newtonian time, and that mathematical tricks involving length contraction and time dilation, known as the Lorentz transformations, could be used to describe what two observers in different inertial frames would observe. Einstein [4] introduced as a postulate "*the principle of relativity*," which states that the equations that represent the fundamental laws of physics such as Maxwell's equations, have the same form in any inertial system. In order to extend Maxwell's equations from one inertial system to all inertial systems, Einstein took Lorentz' mathematical tricks seriously, required the use of the Lorentz transformations, and proposed that time and space were truly interdependent and relative.

Einstein also introduced a second postulate, which states that in empty space, light propagates with a definite velocity that is independent of the state of motion of the emitting body. In his paper, Einstein extended the conjectures of corpuscular mechanics to optical theory, but neglected the wave properties that are equally important in optical phenomena. The neglect of the wave properties led to the conclusion that space and time were relative quantities that depended on the relative velocity of different observers. Inclusion of the wave properties leads to the conclusion that space and time are absolute and it is the wave properties of light such as frequency and wavelength, or angular frequency and angular wave number, and not space and time that depend on the relative velocity of different observers. Here I recast Maxwell's equations for free space in a relativistic form that is consistent with the two postulates of the Special Theory of Relativity. I also show how the anisotropy in the Poynting vectors results in radiation friction that prevents particles with a charge and/or a magnetic moment from going faster than the speed of light.

2. Results and Discussion

The Special Theory of Relativity is founded on the constancy of the speed of light. The speed of light (c) can be defined in two ways. The first way, which relates the speed of light to the electrical permittivity (ε_o) of the vacuum and magnetic permeability (μ_o) of the vacuum, neglects the wave-like properties of light:

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$$c = \frac{1}{\sqrt{\varepsilon_o \mu_o}} \tag{1}$$

The second way to define the speed of light takes the wave-like properties of frequency (ν) and wavelength (λ) or angular frequency (ω) and angular wave number (k) into consideration:

$$c = \nu\lambda = \frac{\omega}{k} \tag{2}$$

The two equations can be combined in a definition of c^2:

$$c^2 = \frac{1}{\sqrt{\varepsilon_o \mu_o}} \frac{\omega}{k} \tag{3}$$

It is the quadratic speed of light that is represented in Maxwell's second-order electromagnetic wave equation.

Eqns. (1), (2) and (3) give definitions of the speed of light that are applicable to any inertial system but are not applicable to systems in uniform motion relative to each other where the wave-like properties of light are best described by the Doppler effect expanded to second order with respect to velocity [5]. The second-order Doppler effect can account for the relativity of simultaneity [5], the optics of moving bodies [6-8], the maximum speed of bodies with a charge and/or magnetic moment [9,10], irreversibility [11] and the inertia of energy [12] without the need to introduce relative and interdependent four-dimensional space-time.

When the emitting body (observer at source) and another observer ($k_{observer}$) are in inertial frames moving relative to each other at velocity v, the square of the vacuum speed of light is given by:

$$c^2 = \frac{1}{\sqrt{\varepsilon_o \mu_o}} \frac{\omega_{source}}{k_{observer}} \frac{\sqrt{1 - \frac{v\cos\theta}{c}}}{\sqrt{1 + \frac{v\cos\theta}{c}}} \tag{4}$$

Where, θ is defined as the angle subtended by a light rays extending from the source to the observer and the velocity vector that ends at the observer (Fig. 1). θ = 0 when the velocity vector and light ray are parallel and $\theta = \pi$ when the velocity vector and light ray are antiparallel.

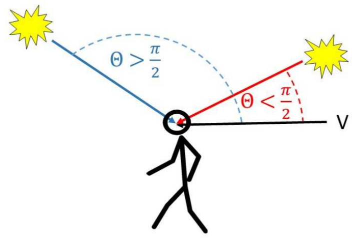

Fig.1: The definition of θ, where θ is defined as the angle subtended by a light rays extending from the source to the observer and the velocity (v) vector that ends at the observer

As the relative velocity increases, $k_{observer}$ decreases when $\theta < \frac{\pi}{2}$ and increases when $\theta > \frac{\pi}{2}$. As a result, c^2 remains constant in any and all inertial frames [5]. Multiplying Eqn. (4) by $1 = \frac{\sqrt{1-\frac{v\cos\theta}{c}}}{\sqrt{1-\frac{v\cos\theta}{c}}}$, we get:

$$c^2 = \frac{1}{\sqrt{\varepsilon_o \mu_o}} \frac{\omega_{source}}{k_{observer}} \frac{\sqrt{1-\frac{v\cos\theta}{c}}}{\sqrt{1+\frac{v\cos\theta}{c}}} \frac{\sqrt{1-\frac{v\cos\theta}{c}}}{\sqrt{1-\frac{v\cos\theta}{c}}} \tag{5}$$

which simplifies to:

$$c^2 = \frac{1}{\sqrt{\varepsilon_o \mu_o}} \frac{\omega_{source}}{k_{observer}} \frac{1-\frac{v\cos\theta}{c}}{\sqrt{1-\frac{v^2 cos^2\theta}{c^2}}} \tag{6}$$

While Eqn. (6) shares similarities with the Lorentz transformation equations, there are fundamental differences. The cosines in both the numerator and the denominator in $\frac{1-\frac{v\cos\theta}{c}}{\sqrt{1-\frac{v^2 cos^2\theta}{c^2}}}$ represent the spatial relationships between the velocity (v) vector and the observed light ray (c). This contrasts with Doppler's principle for relative velocities given by Einstein [4], which is based on the Lorentz transformations. In Einstein's equation, the cosine in the numerator represents the spatial relationship between the observer and the source and the cosine is absent in the denominator, which represents the temporal relationship between the observer and the source. Eqn. (6) reduces to Einstein's Doppler principle for relative velocities when $cos^2\theta = 1$. Einstein's Doppler equation can be viewed mathematically as a limiting case of Eqn. (6).

Eqn. (6) can be used to define the square root of the product of the electric permittivity and the magnetic permeability of the vacuum observed in two inertial systems moving at relative velocity v:

$$\sqrt{\varepsilon_o \mu_o} = \frac{\omega_{source}}{c^2 k_{observer}} \frac{\sqrt{1 - \frac{v\cos\theta}{c}}}{\sqrt{1 + \frac{v\cos\theta}{c}}} \tag{7}$$

The product of the electric permittivity and the magnetic permeability of the vacuum is a constant that determines the magnitude of the Dopplerization of the electromagnetic waves for an observer at rest with the source and an observer moving relative to the source.

The electric permittivity and the magnetic permeability of the vacuum are the only constants in Maxwell's equations. The form of Maxwell's equations for free space that are applicable when the source and the observer are in the same inertial frame are given below:

$$\nabla \cdot E = 0 \tag{8a}$$

$$\nabla \cdot B = 0 \tag{8b}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{8c}$$

$$\nabla \times B = \varepsilon_o \mu_o \frac{\partial E}{\partial t} \tag{8d}$$

According to Eqn. (8c), which is a statement of Faraday's law, the linear displacement of a magnet results in a circular electric field. The induced electric field acts as an electromotive force that can drive a current in a wire. According to Eqn. (8d), which is a statement of the Ampere-Maxwell equation for free space, a linear displacement current transforms an electric field into a circular magnetic field. The Ampere-Maxwell equation contains the constants $\varepsilon_o \mu_o$ that are inversely related to the square of the speed of light.

Because the Ampere-Maxwell equation includes the speed of light, we can ask what the magnetic field would look like to an observer at rest with the source and an observer moving relative to the source. By combining Eqns. (1) and (7), we get $\varepsilon_o \mu_o = \sqrt{\varepsilon_o \mu_o} \frac{k_{observer}}{\omega_{source}} \frac{\sqrt{1 + \frac{v\cos\theta}{c}}}{\sqrt{1 - \frac{v\cos\theta}{c}}}$, which lets us take both the particle-like and wave-like properties of the electric and magnetic fields in Eqn. (8d) into consideration[1]. Eqn. (8d) becomes:

$$\nabla \times B = \sqrt{\varepsilon_o \mu_o} \frac{k_{observer}}{\omega_{source}} \frac{\sqrt{1 + \frac{v\cos\theta}{c}}}{\sqrt{1 - \frac{v\cos\theta}{c}}} \frac{\partial E}{\partial t} =$$

$$\sqrt{\varepsilon_o \mu_o} \frac{k_{observer}}{\omega_{source}} \frac{1 + \frac{v\cos\theta}{c}}{\sqrt{1 - \frac{v^2 cos^2\theta}{c^2}}} \frac{\partial E}{\partial t} \tag{9}$$

After taking the curl of both sides, we get:

$$\nabla \times \nabla \times B = \sqrt{\varepsilon_o \mu_o} \frac{k_{observer}}{\omega_{source}} \frac{1 + \frac{v\cos\theta}{c}}{\sqrt{1 - \frac{v^2 cos^2\theta}{c^2}}} \frac{\partial \nabla \times E}{\partial t} \tag{10}$$

Substituting Eqn. (8c) into Eqn. (10), we get:

$$\nabla \times \nabla \times B = -\sqrt{\varepsilon_o \mu_o} \frac{k_{observer}}{\omega_{source}} \frac{1 + \frac{v\cos\theta}{c}}{\sqrt{1 - \frac{v^2 cos^2\theta}{c^2}}} \frac{\partial^2 B}{\partial t^2} \tag{11}$$

Using the vector identity $\nabla \times \nabla \times B = \nabla(\nabla \cdot B) - \nabla^2 B$, we get:

$$\nabla \times \nabla \times B = \nabla(\nabla \cdot B) - \nabla^2 B =$$

$$-\sqrt{\varepsilon_o \mu_o} \frac{k_{observer}}{\omega_{source}} \frac{1 + \frac{v\cos\theta}{c}}{\sqrt{1 - \frac{v^2 cos^2\theta}{c^2}}} \frac{\partial^2 B}{\partial t^2} \tag{12}$$

Since $\nabla \cdot B = 0$ for free space, after simplifying, we get the second-order wave equation for the magnetic field:

$$\nabla^2 B = \sqrt{\varepsilon_o \mu_o} \frac{k_{observer}}{\omega_{source}} \frac{1 + \frac{v\cos\theta}{c}}{\sqrt{1 - \frac{v^2 cos^2\theta}{c^2}}} \frac{\partial^2 B}{\partial t^2} \tag{13}$$

After rearranging we get:

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[1] Individually, $\sqrt{\mu_o}$, $\sqrt{\varepsilon_o}$, and the impedance of free space (Z_o) are given by:

$$\sqrt{\mu_o} = \frac{k_{observer}}{\sqrt{\varepsilon_o}\,\omega_{source}} \frac{\sqrt{1 + \frac{v\cos\theta}{c}}}{\sqrt{1 - \frac{v\cos\theta}{c}}}$$

$$\sqrt{\varepsilon_o} = \frac{k_{observer}}{\sqrt{\mu_o}\,\omega_{source}} \frac{\sqrt{1 + \frac{v\cos\theta}{c}}}{\sqrt{1 - \frac{v\cos\theta}{c}}}$$

$$Z_o = \sqrt{\frac{\mu_o}{\varepsilon_o}}$$

$$\frac{1}{\sqrt{\varepsilon_o \mu_o}} \frac{\omega_{source}}{k_{observer}} \frac{1 - \frac{v\cos\theta}{c}}{\sqrt{1 - \frac{v^2 cos^2\theta}{c^2}}} \nabla^2 B = \frac{\partial^2 B}{\partial t^2} \qquad (14)$$

Where, $\frac{1}{\sqrt{\varepsilon_o \mu_o}} \frac{\omega_{source}}{k_{observer}} \frac{1 - \frac{v\cos\theta}{c}}{\sqrt{1 - \frac{v^2 cos^2\theta}{c^2}}}$ is a constant equal to c^2 yet includes the relative velocity between the inertial frame of the source and the inertial frame of the observer. When the relative velocity vanishes or $\theta = \pm\frac{\pi}{2}$, Eqn. (14) reduces to Maxwell's wave equation for the magnetic field. Maxwell's wave equation for the magnetic field can be viewed as a limiting case of Eqn. (14), when the relative velocity vanishes.

Starting with Eqn. (8c), we can derive the second-order wave equation for the electric field by taking the curl of both sides.

$$\nabla \times \nabla \times E = -\frac{\partial \nabla \times B}{\partial t} \qquad (15)$$

After substituting Eqn. (9) into Eqn. (15), we get:

$$\nabla \times \nabla \times E = -\sqrt{\varepsilon_o \mu_o} \frac{k_{observer}}{\omega_{source}} \frac{1 + \frac{v\cos\theta}{c}}{\sqrt{1 - \frac{v^2 cos^2\theta}{c^2}}} \frac{\partial^2 E}{\partial t^2} \qquad (16)$$

Using the vector identity $\nabla \times \nabla \times E = \nabla(\nabla \cdot E) - \nabla^2 E$ and assuming that $\nabla \cdot E = 0$ for free space, we get a second-order wave equation for the electric field:

$$\nabla^2 E = \sqrt{\varepsilon_o \mu_o} \frac{k_{observer}}{\omega_{source}} \frac{\sqrt{1 + \frac{v\cos\theta}{c}}}{\sqrt{1 - \frac{v\cos\theta}{c}}} \frac{\partial^2 E}{\partial t^2} =$$

$$\sqrt{\varepsilon_o \mu_o} \frac{k_{observer}}{\omega_{source}} \frac{1 + \frac{v\cos\theta}{c}}{\sqrt{1 - \frac{v^2 cos^2\theta}{c^2}}} \frac{\partial^2 E}{\partial t^2} \qquad (17)$$

After rearranging, we get:

$$\frac{1}{\sqrt{\varepsilon_o \mu_o}} \frac{\omega_{source}}{k_{observer}} \frac{1 - \frac{v\cos\theta}{c}}{\sqrt{1 - \frac{v^2 cos^2\theta}{c^2}}} \nabla^2 E = \frac{\partial^2 E}{\partial t^2} \qquad (18)$$

Where, $\frac{1}{\sqrt{\varepsilon_o \mu_o}} \frac{\omega_{source}}{k_{observer}} \frac{1 - \frac{v\cos\theta}{c}}{\sqrt{1 - \frac{v^2 cos^2\theta}{c^2}}}$ is a constant equal to c^2 yet includes the relative velocity between the inertial frame of the source and the inertial frame of the observer. When the relative velocity vanishes or $\theta = \pm\frac{\pi}{2}$, Eqn. (18) reduces to Maxwell's wave equation for the electric field. Maxwell's wave equation for the electric field can be viewed as a limiting case of Eqn. (18), when the relative velocity vanishes

The essential difference between the transformation equations given above, where relative velocity is introduced into Maxwell's equations and Einstein's transformation equations, where relative velocity is introduced *post hoc* into Maxwell's equations, is whether it is the amplitude of the field or the angular frequency/wave number of the field that is relative. Einstein's [4] transformations shows a velocity-dependent transformation between the amplitudes of the electric field and the magnetic field, where the amplitudes of the fields are relative. In the transformations given here, the amplitudes of the electric and magnetic fields are invariant but there is a speed- and angle-dependent transformation of the angular frequency/wave number of the electric and magnetic fields. This is consistent with the effect of relative motion observed in all other kinds of waves, including water waves and sound waves. That all waves should be treated the same is in the interest of what Ernst Mach [13] called the economy of science.

The velocity-induced change in the wave properties of the electric field and the magnetic field means that even though the amplitudes of the electric and magnetic fields remain constant when the velocity changes, the time-averaged energy densities ($\langle U \rangle$, in J/m^3) of the fields decrease as the velocity increases when $\theta < \frac{\pi}{2}$ and the time-averaged energy densities ($\langle U \rangle$) of the fields increase as the velocity increases when $\theta > \frac{\pi}{2}$. By necessity, the time-averaged radiation pressure ($\langle P \rangle$, in N/m^2), which is equal to one-third of the energy density, decreases when the velocity increases when $\theta < \frac{\pi}{2}$ and increases when the velocity increases when $\theta > \frac{\pi}{2}$:

$$\langle U \rangle = 3\langle P \rangle = \frac{1}{2}\left(\varepsilon_o E^2 + \frac{B^2}{\mu_o}\right) \frac{1 - \frac{v\cos\theta}{c}}{\sqrt{1 - \frac{v^2 cos^2\theta}{c^2}}} \qquad (19)$$

The energy in the radiation field can interact with particles with a charge and/or a magnetic moment. As a result, there is an anisotropy in the radiation pressure on moving particles that have a charge and/or a magnetic moment:

$$\langle P \rangle = \frac{1}{6}\left(\varepsilon_o E^2 + \frac{B^2}{\mu_o}\right) \frac{1 - \frac{v\cos\theta}{c}}{\sqrt{1 - \frac{v^2 cos^2\theta}{c^2}}} \qquad (20)$$

The product of the energy density and the speed of light c gives the power propagating in free space per unit area. Again, by necessity, in free space, the power propagating per unit area decreases when the velocity increases when $\theta < \frac{\pi}{2}$ and increases when the velocity increases when $\theta > \frac{\pi}{2}$. The power propagating per unit area is also known as the Poynting vector ($\langle S\rangle$, in J m^{-2} s^{-1}). Consequently, the Poynting vector increases as the particle with a charge and/or magnetic moment moves towards a source and decreases as the particle moves away:

$$\langle S\rangle = \frac{c}{2}\left(\varepsilon_o E^2 + \frac{B^2}{\mu_o}\right)\frac{1-\frac{v\cos\theta}{c}}{\sqrt{1-\frac{v^2cos^2\theta}{c^2}}} \tag{21}$$

When a particle is at rest with respect to an isotropic radiation field, the Poynting vector is equal in all directions. For a particle moving through the radiation field, that would be isotropic to the particle at rest, the Poynting vectors become anisotropic, being greater in front of the particle and smaller behind the particle:

$$\langle S\rangle = \frac{c}{6}\left(\varepsilon_o E^2 + \frac{B^2}{\mu_o}\right)\frac{1-\frac{v\cos\theta}{c}}{\sqrt{1-\frac{v^2cos^2\theta}{c^2}}} \tag{22}$$

Consequently, the anisotropic Poynting vector resists the movement of any particle that contains a charge and/or a magnetic moment. The Poynting vector that points towards the front of a moving particle approaches infinity as the velocity of the particle approaches the speed of light. The anisotropy of the power per unit area resulting from Dopplerization produces a resistance that limits the velocity of the particle to the speed of light. Here the resistance due to the Doppler effect is presented in terms of the Doppler-shifted electric and magnetic fields. Although the Special Theory of Relativity is based on the assumption of no friction, this resistance is consistent with Einstein concept of radiation friction that he introduced in 1909 [14]. Elsewhere I have presented the resistance due to the second-order Doppler effect in terms of an optomechanical counterforce [9,15], the time rate of change of the magnetic vector potential [10,16], and dark energy [17].

At the onset of the twentieth century, Walter Ritz and Albert Einstein tried to reconcile the fields of mechanics and electromagnetism [18] by uncovering the essential problems that prevented the unification of the two theories. Einstein argued that electromagnetism could be reconciled with mechanics if Maxwell's equations were modified by making space and time relative while keeping the speed of light constant. By contrast, Walter Ritz argued that electromagnetism could be reconciled with mechanics in terms of Euclidean space and Newtonian time if Maxwell's equations were modified by making the speed of light relative and dependent on the velocity of its source. Here I have modified the Ampere-Maxwell equation, and consequently, Maxwell's electromagnetic wave equations, by keeping the speed of light constant but making the angular frequency/wave number of the source and observer relative in a velocity-dependent manner. Unlike Einstein's treatment that makes the amplitudes of the electric and magnetic fields dependent on relative velocity, my treatment keeps the amplitudes of the electric and magnetic fields constant but varies the angular frequency/wave number of the waves that make up the fields.

The Special Theory of Relativity, according to John Norton [19], "*is the fruit of 19th century electrodynamics. It is as much the theory that perfects 19th century electrodynamics as it is the first theory of modern physics. Until this electrodynamics emerged, special relativity could not arise; once it had emerged, special relativity could not be stopped.*"

Here I show that modifying the Ampere-Maxwell equation by expanding the constant to allow for relative motion while still keeping the term constant, I get a relativistically-invariant wave equation that describes the electric and magnetic fields in Euclidean space and Newtonian time as seen by two observers—one at rest with respect to a source and one moving with respect to the source. The wave-equation predicts anisotropy in the frequency/wave number of the radiation that results in an anisotropy of the Poynting vector for a particle moving though a radiation field. Thus the Special Theory of Relativity, which posits waves moving isotropically through a four-dimensional space-time continuum, is sufficient but not necessary to explain the electrodynamic effects of relative motion.

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