Turkish Journal of Physics

http://journals.tubitak.gov.tr/physics/

Research Article

Turk J Phys
(2014) 38: 26 – 38

doi:10.3906/fiz-1302-11

Deriving the Snel–Descartes law for a single photon

Randy WAYNE*
Laboratory of Natural Philosophy, Department of Plant Biology, Cornell University, Ithaca, New York, USA

Received: 28.02.2013 • **Accepted:** 10.07.2013 • **Published Online:** 17.01.2014 • **Printed:** 14.02.2014

Abstract: The laws of reflection and refraction are typically derived using a dispersion relation that assumes that the angular frequency stays the same across a vacuum–dielectric interface and that the angular wave number increases. At the single photon level, this means that the total energy of the photon is conserved but the linear momentum is not. Here I derive the laws of reflection and refraction for a single photon using a dispersion relation that is consistent with the conservation of both total energy and linear momentum. To ensure that both conservation laws are upheld, the optical path length must be considered to be more fundamental than the geometrical path length when reckoning optical distances. As long as the medium is transparent, the optical path length traveled by a photon in a given duration of time is constant. By contrast, the geometrical path length traveled in a given duration of time is inversely proportional to the index of refraction. A photon propagating across a vacuum–dielectric interface is considered to be an indivisible package whose total energy (and angular frequency) and linear momentum (and angular wave number) remain constant, but whose velocity decreases with refractive index as a result of unspecified electromagnetic interactions.

Key words: Conservation, energy, linear momentum, photon, reflection, refraction

"Its general transparency, hardness, unchangeable nature, and varied refractive and dispersive powers, render glass a most important agent in the hands of a philosopher engaged in investigating the nature and properties of light."

Michael Faraday [1]

1. Introduction

Electromagnetic light waves differ from mechanical waves in being self-perpetuating. For example, photons emitted from an incandescent source can propagate for billions of years across the vacuum of space without losing their total energy or linear momentum and without getting "tired" [2]. Photons can even pass through a perfect dielectric unchanged and at the velocity they entered as long as the total energy of the photon is far from the resonance energy of the matter through which it propagates. While it is not surprising when the mechanical wave analogy breaks down when applied to optical phenomena such as black body radiation and the photoelectric effect, it is surprising when this analogy breaks down in processes, including reflection and refraction, described for centuries by physical optics.

Quantum mechanics provides a theory to understand the nature of light and it has been applied to elucidate the mechanism of reflection and refraction in 2 complementary ways. Firstly, Cox and Hubbard [3] and Nightingale [4] used quantum mechanics to derive the laws of reflection and refraction for many photons

*Correspondence: row1@cornell.edu

by assuming that energy and momentum are statistically conserved. Secondly, De Grooth [5] derived the laws of reflection and refraction for a single photon using a path integral approach that took into consideration the contributions of all the possible classical paths that a single photon might take. Here I present a third approach that yields an unanticipated result that challenges a prevailing assumption of the wave theory regarding the boundary conditions at an optical interface.

2. Results

It is generally assumed that at a vacuum–dielectric interface the angular frequency (ω_o) of the incident, reflected, and transmitted light remains constant while the angular wave number ($k_d = \frac{2\pi}{\lambda_d}$) varies according to the dispersion relation given by [6–28; Figure 1]:

$$\frac{\omega_o}{k_d} = v_d = \frac{c}{n_d} \tag{1}$$

where v_d is the velocity of a photon in a dielectric (d) with refractive index (n_d) and c is the vacuum speed of light. The vacuum speed of a photon, the carrier of the electromagnetic force, depends on the electric permittivity (ε_o) and the magnetic permeability (μ_o) of the vacuum:

$$c = \sqrt{\frac{1}{\varepsilon_o \mu_o}}. \tag{2}$$

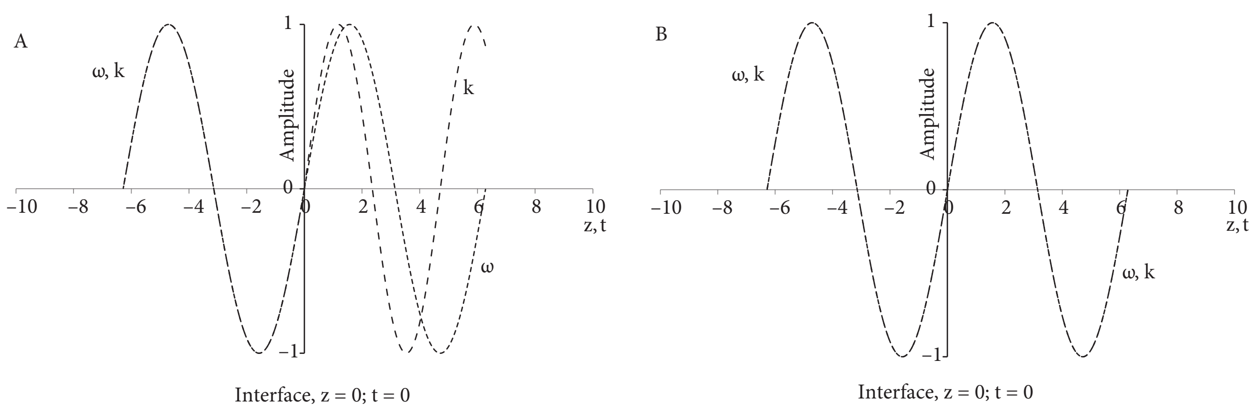

Figure 1. A plot of the function: cos (nkr - ωt), when either $t = 0$ or $z = 0$. (A) Using the dispersion relation $\omega / k_d = v_d$, in geometrical space, the angular frequency and the angular wave number do not have the same phase relationship within a dielectric when the refractive index is greater than unity. Total energy is conserved but linear momentum is not. (B). Using the dispersion relation $\omega_d / k_d = v_d n_d =$ c, the angular frequency and angular wave number have the same phase relationship in optical space in each medium and both total energy and linear momentum are conserved.

Similarly, the speed of a photon in a transparent dielectric medium depends on the electromagnetic characteristics of the dielectric through which the photon propagates, including the electrical permittivity (ε_d) and the magnetic permeability (μ_d), which are wavelength dependent quantities:

$$v_d = \sqrt{\frac{1}{\varepsilon_d \mu_d}}. \tag{3}$$

The ratio of the speed of a photon in a vacuum to the speed of that photon in a dielectric is characterized by the refractive index, which is given by:

$$n_d = \frac{c}{v_d} = \sqrt{\frac{\varepsilon_d \mu_d}{\varepsilon_o \mu_o}}. \tag{4}$$

The refractive index of a dielectric is wavelength dependent; dispersion, or the decomposition of multichromatic light, results [8,29]. Practically speaking, dispersion results in the desirable decomposition of multichromatic light into its components by prisms [30] as well as the undesirable phenomenon of chromatic aberration of lenses [31–33]. Given the standard textbook definitions that the angular frequency is related to the total energy of a photon ($E = \hbar\omega$) and that the angular wave number is related to the linear momentum of a photon ($p = \hbar k$), the dispersion relation given by Eq. (1) demands that the total energy but not the linear momentum be conserved as a photon propagates across a vacuum–dielectric interface.

The mechanical wave theory of light described the propagation of light through an elastic solid aether that permeated all transparent bodies [34–38]. A holdover from this theory is the assumption that the angular wave number varies as the angular frequency of the photon remains constant for a photon propagating across a vacuum–dielectric interface [6–28]. The mechanical wave theory of light held that the velocity (v) of light through the aether depended on the elasticity (e) and density (ρ) of the aether in the propagation medium, where

$$v = \sqrt{\frac{e}{\rho}}. \tag{5}$$

Using the law of conservation of energy, the mechanical wave theory of light held either the elasticity or the density to remain constant across an interface. Augustine-Louis Cauchy, Franz Neumann, and James MacCullagh initially derived the laws of reflection and refraction by assuming that the density but not the elasticity of the luminous aether was equal on both sides of the interface. Later, Augustin-Jean Fresnel derived the same laws by assuming that the elasticity but not the density was equal on both sides of the interface. Fresnel's assumptions led to further explanations in terms of the reflection and refraction of polarized light. The assumption that one property but not the other must remain constant across an interface is not necessary for explaining the laws of reflection and refraction. The assumption used by the electromagnetic theory of light proposes that the angular frequency but not the angular wave number is equal on both sides of a vacuum–dielectric interface [6–28]. I will show below neither assumption is necessary.

At the onset of this derivation, I will only demand that the ratio of the total energy to the linear momentum of a photon remains constant and equal to c, the vacuum speed of light. This means that the ratio of the angular frequency (ω) to the angular wave number (k) in a dielectric is also constrained such that their ratio must also be equal to c. Subsequently, I will invoke both conservation of total energy and conservation of linear momentum for a single photon propagating across a vacuum–dielectric interface. The constant c represents many relations related to light besides the proposed ratio of the angular frequency to the angular wave number of a photon in a dielectric. It also represents the product of the refractive index of a medium and the velocity of light in that medium, the amplitude of the electric field to the magnetic field, and the ratio of the time-averaged Poynting flux to the radiation pressure [39–41]. Given my initial assumption, the relation between the total energy (E_d) and linear momentum (p_d) of a photon and the relation between the angular frequency and the angular wave number of a photon propagating though a medium (d) would be given by:

$$\frac{\omega_d}{k_d} = \frac{\hbar\omega_d}{\hbar k_d} = \frac{E_d}{p_d} = v_d n_d = c, \tag{6}$$

which differs from the dispersion relation given by Eq. (1) in that both k and ω can remain constant across the interface.

According to Eq. (6), the ratio of the total energy of a photon to its linear momentum, like the ratio of its angular frequency to its angular wave number, is constant and independent of the refractive index of the propagation medium. After rearranging Eq. (6), we can solve for the velocity (v_d) of a photon in a dielectric:

$$\frac{E_d}{n_d p_d} = \frac{\hbar\omega_d}{n_d \hbar k_d} = v_d. \tag{7}$$

Eq. (6) is a dispersion relation that differs from Eq. (1) in that both k and ω can remain constant across the interface, and $v_d n_d$ is a constant on both sides of the interface and equal to c. The velocity of a photon in a dielectric depends on the wavelength-dependent refractive index. Given that this velocity depends on the wavelength of the photon, it is similar to the phase velocity. Likewise $v_d n_d$ is similar to the group velocity in that $v_d n_d$ and the group velocity are both independent of wavelength [42]. As we will see below, the phase velocity relates to geometrical space while $v_d n_d$ relates to optical space. To assert that the total energy and the total linear momentum must both be conserved implies:

$$E_d = E_o \tag{8a}$$

$$\hbar\omega_d = \hbar\omega_o \tag{8b}$$

$$\omega_d = \omega_o \tag{8c}$$

and

$$p_d = p_o \tag{9a}$$

$$\hbar k_d = \hbar k_o \tag{9b}$$

$$k_d = k_o \tag{9c}$$

and we see that both the angular wave number and the angular frequency remain constant across a vacuum–dielectric interface (Figure 1). I now use the dispersion relation defined in Eq. (6) to derive the laws of reflection and refraction.

The electromagnetic wave properties of a photon propagating through a medium with wavelength-dependent refractive index (n_d) can be described by the following equation:

$$\Psi = \Psi_{\max} \cos(\vec{k} \cdot n_d \vec{r} - \omega t), \tag{10}$$

where $n_d r$ is the optical path length (OPL); $\vec{k}$ ($= ka\hat{x} + kb\hat{z}$) is the angular wave vector for a photon with angular wave number (k), where a and b are the directional cosines in the x and z directions, respectively; $\hat{x}$ is the unit vector parallel to a vacuum–dielectric interface and $\hat{z}$ is the unit vector normal to a vacuum–dielectric interface, pointing away from the incident medium (Figure 2). The direction of propagation, $n_d\vec{r}$ ($= n_d x\hat{x} + n_d z\hat{z}$), is assumed to be parallel to $\vec{k}$. The dot product, $\vec{k} \cdot n_d \vec{r} = k(a n_d x \hat{x}\hat{x} + b n_d z \hat{z}\hat{z}) = k(a n_d x + b n_d z)$, where $a = \cos\alpha$, $b = \cos\beta$. Ψ represents any of the standard electromagnetic fields ($\vec{E}$, $\vec{D}$, $\vec{B}$, $\vec{H}$) or potentials (φ, $\vec{A}$) and $\Psi_{\max}$ is the maximal value of the field or potential. The velocity ($\vec{v}_d$) of the photon of angular wave number (k) is given by:

$$\frac{\vec{r}}{t} = \vec{v}_d = \frac{\vec{c}}{n_d} = \frac{E}{n_d \vec{p}}, \tag{11}$$

where $\frac{\vec{r}}{t}$ is the velocity of the photon through geometrical space and $\frac{n_d\vec{r}}{t}$ is the velocity of the photon through optical space. Assuming that both the linear momentum relative to the propagation direction of the photon and the total energy are conserved and constant for a photon propagating through any dielectric medium, substitute $\frac{\vec{p}}{\hbar}$ for $\vec{k}$, and $\frac{E}{\hbar}$ for ω to get:

$$\Psi = \Psi_{\max} \cos\left(\frac{\vec{p}}{\hbar} \cdot n_d \vec{r} - \frac{E}{\hbar} t\right). \tag{12}$$

At any point (x) on the interface ($z = 0$), the photon can be described by 3 separate equations where the subscripts i, r, and t indicate that the photon is propagating through the incident medium with refractive index (n_i), the reflection medium with refractive index (n_r), or the transmission medium with refractive index (n_t):

$$\Psi_i = \Psi_{\max,i} \cos\left(\left(\frac{\vec{p}}{\hbar}\right)_i \cdot n_i \vec{r} - \left(\frac{E}{\hbar}\right)_i t\right) \tag{13}$$

$$\Psi_r = \Psi_{\max,r} \cos\left(\left(\frac{\vec{p}}{\hbar}\right)_r \cdot n_r \vec{r} - \left(\frac{E}{\hbar}\right)_r t\right) \tag{14}$$

$$\Psi_t = \Psi_{\max,t} \cos\left(\left(\frac{\vec{p}}{\hbar}\right)_t \cdot n_t \vec{r} - \left(\frac{E}{\hbar}\right)_t t\right) \tag{15}$$

Eqs. (11), (12), and (13) describe the wavelength-dependent properties of the photon when it is incident, reflected, or transmitted, respectively.

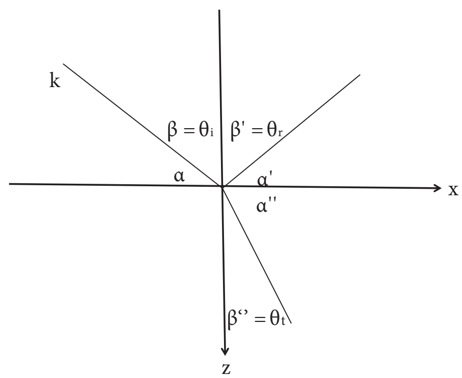

Figure 2. A Cartesian coordinate system describing reflection and refraction. The plane of incidence is given by the x and z axes. The directional cosines relative to the x axis are given by α, α', and α''; and the directional cosines relative to the z axis are given by β, β', and β''. The angles of incidence, reflection, and transmission are given by θ_i, θ_r, and θ_t, respectively.

For a perfect transparent dielectric that does not act as a source or a sink for photons, the boundary conditions are such that the total tangential component of Ψ on one side of the interface is equal to the total tangential component of Ψ on the other side of the interface, according to the following equation:

$$\Psi_{\max,i} \cos\left(\left(\frac{\vec{p}}{\hbar}\right)_i \cdot n_i \vec{r} - \left(\frac{E}{\hbar}\right)_i t\right) + \Psi_{\max,r} \cos\left(\left(\frac{\vec{p}}{\hbar}\right)_r \cdot n_r \vec{r} - \left(\frac{E}{\hbar}\right)_r t\right) = \Psi_{\max,t} \cos\left(\left(\frac{\vec{p}}{\hbar}\right)_t \cdot n_t \vec{r} - \left(\frac{E}{\hbar}\right)_t t\right). \tag{16}$$

In order to ensure continuity across the interface where $z = 0$, Ψ_i, Ψ_r, and Ψ_t must have the same functional dependence on $\vec{r}$ and t together. Thus the arguments of the cosines, but not necessarily just the angular frequencies must be equal [43-47]. Setting the arguments of the cosines to be equal, we get:

$$\left(\frac{\vec{p}}{\hbar}\right)_i \cdot n_i \vec{r} - \left(\frac{E}{\hbar}\right)_i t = n_r \left(\frac{\vec{p}}{\hbar}\right)_r \cdot n_r \vec{r} - \left(\frac{E}{\hbar}\right)_r t = \left(\frac{\vec{p}}{\hbar}\right)_t \cdot n_t \vec{r} - \left(\frac{E}{\hbar}\right)_t t. \tag{17}$$

After cancelling the reduced Planck's constant, we get:

$$\vec{p}_i \cdot n_i\vec{r} - E_i t = \vec{p}_r \cdot n_r\vec{r} - E_r t = \vec{p}_t \cdot n_t\vec{r} - E_t t. \tag{18}$$

An incident photon can be either reflected or refracted at the interface of a perfect dielectric. A photon that is going to be reflected at the vacuum–dielectric interface must satisfy:

$$p_i \cdot n_i\vec{r} - E_i t = p_r \cdot n_r\vec{r} - E_r t. \tag{19}$$

After rearranging, we get:

$$\vec{p}_i \cdot n_i\vec{r} - \vec{p}_r \cdot n_r\vec{r} = E_i t - E_r t. \tag{20}$$

At any time, including $t = 0$ [48,49], which is defined as a unique instant, which has a duration of approximately 10^{-15} s for visible light when the photon straddles the interface, and which is the only time when the incident photon and the reflected photon are indistinguishable, we get:

$$\vec{p}_i \cdot n_i\vec{r} = \vec{p}_r \cdot n_r\vec{r}. \tag{21}$$

When $z = 0$ for any x, then $\vec{p}{\cdot}n\vec{r} = pnx\cos\alpha + pnz\cos\beta = pnx\cos\alpha$ where $\cos\alpha$ is a directional cosine. If the angle of incidence (θ_i) is a complementary angle to α and the angle of reflection (θ_r) is a complementary angle to α', then $\cos\alpha = \sin\theta_i$ and $\cos\alpha' = \sin\theta_r$. Since $n_i = n_r$, $p_i = p_r$, and $x = x$ we get:

$$\sin\theta_i = sin\theta_r. \tag{22}$$

Assuming conservation of linear momentum, after cancelling, we get:

$$\theta_i = \theta_r, \tag{23}$$

which is the law of reflection.

For a perfect dielectric, a photon that is going to be transmitted at the vacuum–dielectric interface must satisfy:

$$\vec{p}_i \cdot n_i\vec{r} - E_i t = \vec{p}_t \cdot n_t\vec{r} - E_t t. \tag{24}$$

After rearranging we get,

$$\vec{p}_i \cdot n_i\vec{r} - \vec{p}_t \cdot n_t\vec{r} = E_i t - E_t t. \tag{25}$$

At any time, including $t = 0$ [48,49], which is defined as a unique instant, which has a duration of approximately 10^{-15} s for visible light when the photon straddles the interface, and which is the only time when the incident photon and the transmitted photon are indistinguishable, we get:

$$\vec{p}_i \cdot n_i\vec{r} - \vec{p}_t \cdot n_t\vec{r}. \tag{26}$$

When $z = 0$ for any x, then $\vec{p}{\cdot}n\vec{r} = npx\,cos\alpha + npz\cos\beta = pnx\,cos\alpha$. If the angle of incidence (θ_i) is a complementary angle to α and the angle of transmission (θ_t) is a complementary angle to α'', then $\cos\alpha = \sin\theta_i$ and $\cos\alpha'' = \sin\theta_t$. Assuming conservation of linear momentum and after cancelling, we get:

$$n_i\sin\theta_i = n_t\,sin\theta_t \tag{27}$$

which is the Snel–Descartes law for the refraction of light by a dielectric [50].

I derived the laws of reflection and refraction by assuming that both the angular frequency and the angular wave number remain constant when a photon propagates though optical space across a vacuum–dielectric interface. With this view, a dielectric decomposes or disperses multi-chromatic light by color, but does not change the color. Color is equally characterized in optical space by its angular frequency, frequency, angular wave number, and wavelength.

3. Discussion

The eye can register the absorption of a single photon [51–53] and according to Cox and Hubbard [3] it is assumed that the photons of visible light "occupy volumes large in comparison with intra-atomic distances, so that many electrons may be included in the volume occupied by one quantum" as opposed to "compact quanta of large energy, such as those considered as acting in the Compton effect." What happens to a single photon of visible light before it is irreversibly absorbed by the pigments of the eye is of interest to biophysicists. I propose a derivation of the laws of reflection and refraction for a single photon by first constraining the ratio of the total energy and the linear momentum of a photon to be equal to the product of the velocity of a photon in the medium and the refractive index of that medium. This product remains constant as the photon propagates across an interface. I then require that the total energy and the linear momentum of the photon both be conserved. In this view, the angular frequency, angular wave number, total energy, and linear momentum of a photon change neither as it crosses a vacuum–dielectric interface nor as it crosses the dielectric–vacuum interface. The exiting photon is indistinguishable from the entering photon, and the perfect dielectric is not displaced. Furthermore, the total energy of a photon propagating through the vacuum or through a dielectric is always related to its initial angular frequency (ω), frequency (ν), angular wave number (k), and wavelength (λ) and can be given in the following equivalent ways:

$$E = \hbar\omega = h\nu = \hbar kc = \frac{hc}{\lambda}. \tag{28}$$

If Eq. (28) is true, the question of whether a light meter measures a frequency-dependent aspect of light or a wavelength-dependent aspect becomes moot. A light meter will register any of the above quantities equally and interchangeably.

Likewise the linear momentum (p) of a photon propagating through the vacuum or a dielectric is related to its initial angular frequency, frequency, angular wave number, and wavelength and can be given in the following equivalent ways:

$$p = \frac{\hbar\omega}{c} = \frac{h\nu}{c} = \hbar k = \frac{h}{\lambda}. \tag{29}$$

These relations require one to consider the propagation of a photon through optical space as opposed to geometrical space. By establishing a dispersion relation that explicitly takes into consideration the refractive index and thus describes the propagation of a photon through optical space as opposed to geometrical space, I have been able to show that it is possible that both the angular frequency and the angular wave number can either remain constant as a photon propagates across an interface (when demanding conservation of both total energy and linear momentum) or can change in unison. The second alternative is consistent with the dispersion relation that gives rise to the relativistic Doppler effect where both the angular frequency and the angular wave number change in unison with relative velocity [54,55].

The assumption that both the total energy and linear momentum of a photon are conserved as it crosses a dielectric interface, and its consequence, that both the frequency and the wavelength of a photon do not change as it propagates across an interface through optical space, is helpful for photobiologists. The existing lack of clarity has been captured by Sönke Johnsen [56] in *The Optics of Light. A Biologist's Guide to Light in Nature*, when he discusses the question of frequency versus wavelength. Is the color of light determined by its frequency or its wavelength? He writes, "So suppose that a beam of sunlight goes from air ($n \cong 1$) into the ocean ($n \cong 1.33$). The index goes up, so the phase velocity drops by a factor of 1.33. Since this is the product of the frequency and wavelength, one of the two (or both) has to also drop. It turns out that the frequency stays the same and the wavelength drops. In this case, a "green" 550 nm photon actually has a wavelength of 414 nm in the ocean. So frequency seems to be more fundamental than wavelength. Also, remember the energy of a photon is proportional to frequency, but not to wavelength....This is important, because in many processes, such as absorption, it is the energy of the photon that matters, not its wavelength. For example, even though the wavelength of a "green" photon inside our eye depends on whether the eye is full of water or air, our perception of it doesn't change, because absorption of light by photoreceptors depends on the energy of the photons, which is related to the unchanging frequency."

Assuming that the conservation of linear momentum is a foundational postulate of physical optics, then one must substitute the optical path length for the geometrical path length when reckoning distances. There is something unexpected about optical space. In optical space, c is always a constant since $v_d n_d$ is always equal to the vacuum speed of light. The constancy of the optical path length between the object and the image is essential for the design and use of chromatic aberration-free optical instruments. Unfortunately, in the literature, there is confusion whether to use the vacuum wavelength or the assumed wavelength in the dielectric and whether to use the geometrical path length or the optical path length when one is quantifying the optical path in a microscope [57–62]. If one assumes that total energy but not linear momentum is conserved across an interface, the frequency of vibration ($\nu_o = \nu_d$) remains constant but the wavelength decreases ($\lambda_d = \frac{\lambda_o}{n_d}$), and one must use the geometrical distance (L_d) and not the optical path length (n_dL) to determine the number of wave fronts (N):

$$N = \frac{L_d}{\lambda_d}. \tag{30}$$

According to Slayter [58], as "a consequence of the lowered velocity and invariant frequency of light in media... more cycles of vibration are 'squeezed' into a given path length in media than occur along the same path length in air."

By contrast, if one assumes that both total energy and linear momentum are conserved, then one must use the optical path length (OPL $= n_d L_d$) and the incident wavelength of the photon in the vacuum to determine the number of wave fronts (N):

$$N = \frac{n_d L_d}{\lambda_o}. \tag{31}$$

Since $\lambda_d = \frac{\lambda_o}{n_d}$, Eqs. (30) and (31) are equivalent and give the correct answer. However, one must be careful not to interchange the 2 descriptions of lengths and the 2 descriptions of wavelength. According to Eq. (33), the wavelength of the photon, along with its frequency, total energy, and linear momentum, does not change, but the optical path length is longer than the geometrical path length by the factor n_d. Thus the time it takes for a photon to travel a given geometrical distance through a dielectric whose refractive index is greater than unity is increased and its velocity is reduced (Figure 3).

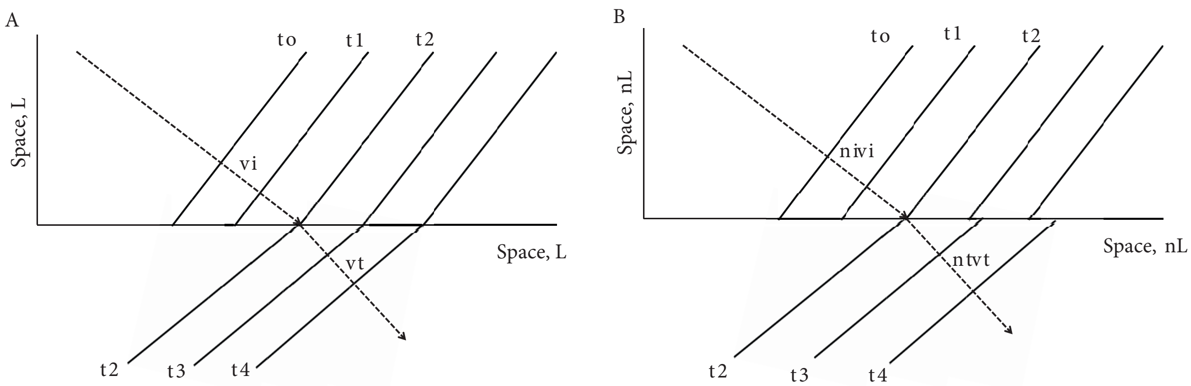

Figure 3. The distance between parallel lines represents the distance traveled by a photon in equal time intervals (dt), i.e. the velocity through A) geometrical space (L) and B) optical space (nL). The optical space is equal to the product of the velocity (v) of the photon propagating through that space and the refractive index (n) of that space. For a vacuum–dielectric interface, the optical space above the horizontal line is not equal to the optical space below the line, but differs by a factor of n. The number of reduced wavelengths that "fit" in geometrical space is equal to the number of initial wavelengths that "fit" in optical space.

If one were to assume that the linear momentum of a photon was not conserved as it propagated across an interface, then one would use the geometrical length, the dispersion relation given by Eq. (1), and the velocity of the photon in each segment of the optical path to determine the transit time for a photon as it propagates along any given path:

$$dt = \sum \frac{L_d}{v_d} = \int \frac{1}{v}\, dL. \tag{32}$$

In fact, to calculate the transit time of a photon [63], opticians typically use optical path length, which, I have shown above, assumes that the wavelength of the photon is constant, and that the transit time is equal to the sum of, or integral of, the optical path lengths divided by the vacuum speed of light:

$$dt = \sum \frac{n_d L_d}{c} = \frac{1}{c} \sum n_d L_d = \frac{1}{c}\ \text{OPL}\ = \frac{1}{c} \int n\, dL. \tag{33}$$

The actual path taken by a photon as it crosses a vacuum–dielectric interface is given by Fermat's principle of least time. While Eqs. (32) and (33) are equivalent, it is interesting that opticians have found success in using the optical path length and the implied constant wavelength of the incident light. This success supports the fundamental nature of conservation of linear momentum for a photon propagating across a vacuum–dielectric interface.

The vacuum speed of light is a fundamental constant of nature [39–41]. It is also, according to Eq. (33), equal to the optical path length traveled by a photon with constant total energy and constant linear momentum in a given duration of time. This is true whether a photon is propagating through the vacuum or any perfectly transparent dielectric. This seems like a foundational principle for describing the propagation of light through dielectrics—and makes it unnecessary to consider the change in wavelength that is predicted from using conservation of total energy without conservation of linear momentum. By utilizing these conservation principles, Eq. (34) gives the distance (dL) a self-perpetuating photon [61] with constant total energy ($E = \hbar\omega$) and linear momentum ($p = \hbar k$) travels through any perfectly transparent medium with refractive index n_d in a given duration of time (dt).

$$dL = \frac{1}{n_d} cdt. \tag{34}$$

Thus the refractive index can be seen to influence the distance a photon travels in a given time without influencing its angular frequency or angular wave number. Since a photon exits a dielectric at the same velocity that it enters, the refractive index can also be seen to reversibly influence the velocity of the photon. This is an extraordinary and unique characteristic of photons since any other particle would be irreversibly slowed down by a medium that altered its velocity. It goes to the question of the nature of the force that causes photons to move [61].

Since light entering and exiting a perfectly transparent dielectric is indistinguishable, I have assumed that the total energy and the linear momentum of a photon do not change as it passes through a perfect dielectric. Consequently, I have arrived at the conclusion that the refractive index slows down a photon without changing its total energy or linear momentum. While there is agreement that the total energy does not change, the characterization of the linear momentum of a photon in a dielectric has been elusive [64–71]. The question becomes does the refractive index stand on its own or does it obtrude some way into the linear momentum term? If a photon is considered to have an unchanging energy but a slower speed in a dielectric, then its linear momentum, known as its Abraham momentum or kinetic momentum ($p_d = \frac{p}{n_d}$ and $\frac{E_d}{p_d} = cn_d$), is predicted to be smaller in the dielectric than in the vacuum (p). On the other hand, if the photon is considered to have an unchanging energy, but a greater angular wave number in a dielectric, then its linear momentum, known as its Minkowski momentum or canonical momentum ($p_d = pn_d$ and $\frac{E_d}{p_d} = \frac{c}{n_d}$), is predicted to be greater in the dielectric than in the vacuum. While the experimental evidence currently supports either definition of linear momentum, perhaps we can say the geometrical average supports conservation of linear momentum for a photon propagating through a perfectly transparent dielectric.

The analysis and interpretation of linear momentum is important for light microscopists who use laser tweezers and it is not inconsequential to consider whether the linear momentum of a photon passing through a perfect dielectric is given by p, pn_d, or p/n_d. Laser tweezers rely on the linear momentum of photons to move stationary or to hold moving microscopic objects [72–80]. The force exerted on an object by a laser is a function of the light intensity, the area of the object, and a dimensionless number (η) that is equal to 2, for perfectly reflecting objects, 1 for perfectly absorbing objects, and according to the above discussion, 0 for perfectly transparent objects. Biological samples although transparent are never perfectly transparent dielectrics because biological macromolecules as well as the water that composes much of them and the water that surrounds them are flexible enough to transform a significant portion of the incident radiant energy into kinetic (translational, vibrational, and rotational) energy. The radiant energy transformed into molecular kinetic energy by the surrounding medium [81,82] could result in a Brownian motion-induced asymmetrical force that should be taken into consideration when calculating the gradient forces exerted on imperfect micrometer-sized dielectric objects. Successful use of optical tweezers results from taking thermal energy into consideration when choosing wavelengths and intensities that minimize optical damage [73] or opticution [76,79].

The laws of reflection and refraction are typically derived by assuming that the angular frequency stays the same across an interface and that the angular wave number increases. Surprisingly, I found that the laws of reflection and refraction can be derived, contrary to the consensus [6–28], if neither the angular frequency nor the angular wave number of a photon changes upon entering and exiting a perfectly transparent dielectric. This is in keeping with conservation of total energy and conservation of linear momentum. This treatment elevates the importance of the optical path length over the geometrical path length in reckoning optical distances and leads directly to the robust equation used by opticians for the transit time of photons in the design of optical systems. This equation is also used by natural scientists for understanding the physical basis of the iridescent colorations of birds and butterflies [83–89]. In hindsight the constancy of angular frequency, angular wave number, frequency, and wavelength seems reasonable since a photon entering a dielectric is indistinguishable from a photon exiting a dielectric.

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